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HIGHER-ORDER SHEAR DEFORMABLE THEORIES FOR FLEXURE OF SANDWICH PLATES—FINITE ELEMENT EVALUATIONS

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Abstract—A simple isoparametric finite element formulation based on a higher-order displacement model for flexure analysis of multilayer symmetric sandwich plates is presented. The assumed displacement model accounts for non-linear variation of inplane displacements and constant variation of transverse displacement through the plate thickness. Further, the present formulation does not require the fictitious shear correction coefficient(s) generally associated with the first-order shear deformable theories. Two sandwich plate theories are developed; one, in which the free shear stress conditions on the top and bottom bounding planes are imposed and another, in which such conditions are not imposed. The validity of the present development(s) is established through, numerical evaluations for deflections/stresses/stress-resultants and their comparisons with the available three-dimensional analyses/closed-form/other finite element solutions. Comparison of results from thin plate, Mindlin and present analyses with the exact three-dimensional analyses yields some important conclusions regarding the effects of the assumptions made in the CPT and Mindlin type theories. The comparative study further establishes the necessity of a higher-order shear deformable theory incorporating warping of the cross-section particularly for sandwich plates.

1. INTRODUCTION

A multilayer sandwich plate is a special form of advanced fibre-reinforced composite laminate. The literature available in the field of laminated composite plates is enormous and the relevant available literature concerning bending stress analysis has been published recently (Kant and Pandya, 1987). We examine here the available literature specifically relevant to the bending problems of sandwich plates.

Reissner (1948) formulated the small deflection theory for the bending of isotropic sandwich type structures. Since this initial publication, a number of papers have been published on various aspects of sandwich bending theory. Kao (1965) developed the governing differential equations for the non-rotationally symmetrical bending of isotropic circular sandwich plates by means of a variational theorem. The governing equations for an orthotropic clamped sandwich plate are derived using the variational principle of minimum potential energy by Folie (1970). The most important contributions were from Srinivas and Rao (1970) and Pagano (1970), who presented exact three-dimensional elasticity solutions for laminated composite/sandwich plates. Whitney (1972) presented a theory analogous to Mindlin's (1951) first-order shear deformation theory for stress analysis of laminated composite/sandwich plates. Later, Lo *et al.* (1977), Murthy (1981), Reddy (1984) and Murty (1985) presented analytical solutions for laminated plate problems using higher-order theories. These theories include warping of the transverse cross-sections. However, they have not presented sandwich plate problems where the effect of warping of the cross-section is predominant. These analytical solutions are limited to a few simple geometries, loading and boundary conditions. This limitation is overcome by adopting the finite element method as a generalized numerical solution technique for practical laminated/sandwich plate problems.

Monforton and Schmit (1969) presented displacement based finite element solutions for sandwich plates using 16 degrees of freedom, 4 noded rectangular elements. Martin

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(1967) adopted 9 degrees of freedom, 3 noded triangular elements with assumed displacement fields. Cook (1972) developed a 12 degrees of freedom, 4 noded general quadrilateral element including transverse shear deformation. Finite element solutions for multilayer sandwich plates have also been presented by Khatua and Cheung (1972, 1973) using triangular and rectangular plate bending elements. Their formulation considered the ideal type of sandwich construction in which the core layers contribute only to the shear rigidity of the plate. Fazio and Ha (1974) presented finite element solutions by explicit derivation of stiffness matrices for bending and membrane actions of a rectangular three layer sandwich plate element using the assumed stress distribution approach. Mawenya and Davies (1974) presented a general formulation for an 8 noded quadratic, isoparametric, multilayer plate bending element which permits the layers to deform locally and incorporates the effects of transverse shear deformation in each layer. Hinton *et al.* (1975), Reddy and Chao (1981) and Putcha and Reddy (1984) adopted assumed displacement, penalty function and mixed methods, respectively, to develop the finite element formulations. Kant and Sahani (1985) presented a displacement based finite element formulation using a 9 noded Lagrangian/Heterosis element. These formulations were based on a first-order shear deformable theory (FOST) which is based on the assumption of the constant shear strain distribution through the laminate thickness and requires the use of shear correction coefficients. Recently, Phan and Reddy (1985), Putcha and Reddy (1986) and Ren and Hinton (1986) presented various finite element formulations of a higher-order theory for laminated plates. However, they have not applied it to sandwich plate problems.

The motivation for the present development comes from the work of Kant (1982) and Kant *et al.* (1982), which was limited to thick isotropic plates. Pandya and Kant (1987, 1988a–c) and Kant and Pandya (1988a,b) extended these developments for orthotropic and laminated composite/sandwich plates. This paper specifically deals with the development and application of a C^0 isoparametric finite element for bending analysis of multilayer symmetric sandwich plates by assuming a higher-order displacement model hitherto not considered. The theory leads to a realistic (parabolic) variation of transverse shear stresses through the plate thickness. It is applicable to an n-layered sandwich plate with $[(n+1)/2]$ stiff layers and $[(n-1)/2]$ alternating weak cores. The 9 noded Lagrangian quadratic element developed has 5 degrees of freedom per node.

2. THEORY

The present higher-order shear deformation theory for symmetric sandwich/laminated plates has been developed by assuming the displacement field in the following form:

$$\begin{aligned} u(x,y,z) &= z\theta_x(x,y) + z^3\theta_x^*(x,y) \\ v(x,y,z) &= z\theta_y(x,y) + z^3\theta_y^*(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

in which w_0 represents the transverse displacement of the midplane and θ_x, θ_y are the rotations of normals to the midplane about the y- and x-axes, respectively, as shown in Fig. 1. The parameters θ_x^*, θ_y^* are the higher-order terms accounting for the flexural mode of deformation in the Taylor series expansion and are also defined at the midplane. The conditions that the transverse shear stresses vanish on the top and bottom faces of the plate are equivalent to the requirement that the corresponding strains be zero on these surfaces. The transverse shear strains are given by

$$\begin{aligned} \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \theta_y + 3z^2\theta_y^* + \frac{\partial w_0}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_x + 3z^2\theta_x^* + \frac{\partial w_0}{\partial x}. \end{aligned} \tag{2}$$

Equating $\gamma_{yz}(x, y, \pm h/2)$ and $\gamma_{xz}(x, y, \pm h/2)$ to zero, we obtain

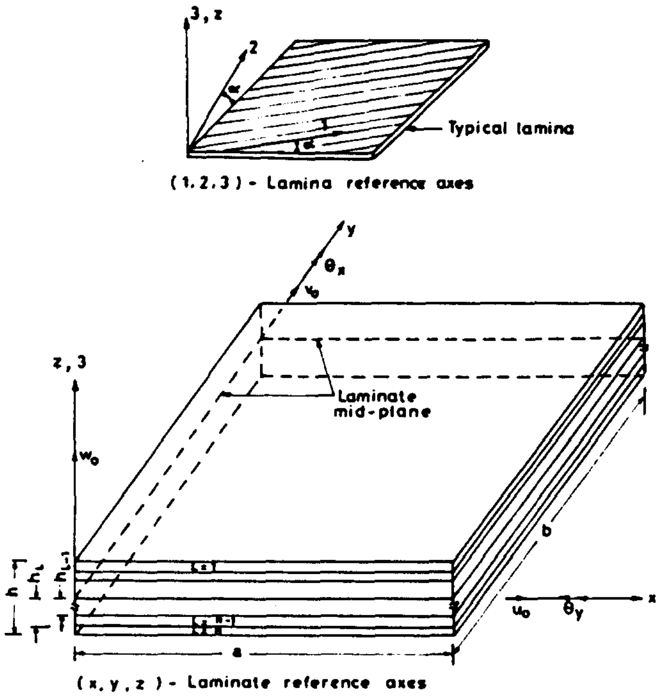

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

$$\theta_y^* = -\frac{4}{3h^2}\left(\theta_y + \frac{\partial w_0}{\partial y}\right);\ \theta_x^* = -\frac{4}{3h^2}\left(\theta_x + \frac{\partial w_0}{\partial x}\right). \tag{3}$$

Murthy (1981) and more recently Reddy (1984) used conditions (3) to eliminate θ_x^* and θ_y^* from the displacement field, which contains additional inplane degrees of freedom (u_0, v_0). In the present theory, we proceed with the displacement field given by eqns (1) and conditions (3) are introduced later in the shear rigidity matrix.

By substitution of eqns (1) in the strain displacement equations of the classical theory of elasticity, the following relationships are obtained:

$$\left[\begin{array}{c|c} \varepsilon_x & \gamma_{xy} \\ \varepsilon_y & \gamma_{yz} \\ \varepsilon_z & \gamma_{xz} \end{array}\right] = \left[\begin{array}{c|c} z\chi_x + z^3\chi_x^* & z\chi_{xy} + z^3\chi_{xy}^* \\ z\chi_y + z^3\chi_y^* & \phi_y + z^2\phi_y^* \\ 0 & \phi_x + z^2\phi_x^* \end{array}\right] \tag{4}$$

in which

$$[\chi_x, \chi_y, \chi_{xy}] = \left[\frac{\partial\theta_x}{\partial x}, \frac{\partial\theta_y}{\partial y}, \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right]$$

$$[\chi_x^*, \chi_y^*, \chi_{xy}^*] = \left[\frac{\partial\theta_x^*}{\partial x}, \frac{\partial\theta_y^*}{\partial y}, \frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\right]$$

$$[\phi_x, \phi_y, \phi_x^*, \phi_y^*] = \left[\frac{\partial w_0}{\partial x} + \theta_x, \frac{\partial w_0}{\partial y} + \theta_y, 3\theta_x^*, 3\theta_y^*\right]. \tag{5}$$

The material constitutive relations for the Lth layer can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^L$$

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{6}$$

where $(\sigma_1, \sigma_2, \tau_{12}, \tau_{23}, \tau_{13})$ are the stress and $(\varepsilon_1, \varepsilon_2, \gamma_{12}, \gamma_{23}, \gamma_{13})$ the linear strain components referred to the lamina coordinate axes (1, 2, 3) as shown in Fig. 1 and C_{ij}'s the reduced material stiffnesses of the Lth lamina and the following relations hold between these and the engineering elastic constants:

$$C_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}; \quad C_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}; \quad C_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$C_{33} = G_{12}, \quad C_{44} = G_{23}, \quad C_{55} = G_{13}. \tag{7}$$

The stress–strain relation for the Lth lamina in the laminate coordinate axes (x, y, z) are written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^L$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \tag{8}$$

in which

$$\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz}\}^t$$

and

$$\boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\}^t \tag{9}$$

are the stress and linear strain vectors with reference to the laminate axes and Q_{ij}'s are the transformed reduced elastic coefficients in the plate (laminate) axes of the Lth lamina. The transformation of the stresses/strains between the lamina and the laminate coordinate systems follows the usual transformation rule given in Jones (1975).

The total potential energy π of the plate is given by

$$\pi = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^t \boldsymbol{\sigma}\,\mathrm{d}V - \int_A \boldsymbol{\delta}^t \mathbf{F}\,\mathrm{d}A \tag{10}$$

in which A is the mid-surface area of the plate, V the plate volume, F the intensity of the force vector corresponding to the degrees of freedom $\boldsymbol{\delta}$ defined as

$$\boldsymbol{\delta} = \{w_0, \theta_x, \theta_y, \theta_x^*, \theta_y^*\}^t. \tag{11}$$

The expressions for the strain components given by relations (4) are substituted in expression (10). The functional given by expression (10) is then minimized while carrying out explicit integration through the plate thickness. This leads to the following ten stress-resultants for the n-layered laminate:

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [z \mid z^3]\,\mathrm{d}z$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [1 \mid z^2]\,\mathrm{d}z. \tag{12}$$

After integration, these relations are written in a matrix form which defines the stress-resultant/strain relations of the laminate and is given by

$$\begin{Bmatrix} \mathbf{M} \\ \mathbf{M}^* \\ --- \\ \mathbf{Q} \\ \mathbf{Q}^* \end{Bmatrix} = \left[\begin{array}{c|c} \mathscr{D}_b & 0 \\ \hline 0 & \mathscr{D}_s \end{array}\right] \begin{Bmatrix} \boldsymbol{\chi} \\ \boldsymbol{\chi}^* \\ --- \\ \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^* \end{Bmatrix}$$

or

$$\bar{\sigma} = \mathscr{D}\bar{\varepsilon} \tag{13}$$

$$\begin{aligned} \mathbf{M} &= \{M_x, M_y, M_{xy}\}^t; & \boldsymbol{\chi} &= \{\chi_x, \chi_y, \chi_{xy}\}^t \\ \mathbf{M}^* &= \{M_x^*, M_y^*, M_{xy}^*\}^t; & \boldsymbol{\chi}^* &= \{\chi_x^*, \chi_y^*, \chi_{xy}^*\}^t \\ \mathbf{Q} &= \{Q_x, Q_y\}^t; & \boldsymbol{\Phi} &= \{\Phi_x, \Phi_y\}^t \\ \mathbf{Q}^* &= \{Q_x^*, Q_y^*\}^t; & \boldsymbol{\Phi}^* &= \{\Phi_x^*, \Phi_y^*\}^t \end{aligned} \tag{14}$$

$$\mathscr{D}_b = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ & Q_{22}H_3 & Q_{23}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_5 \\ & & Q_{33}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_5 \\ & & & Q_{11}H_7 & Q_{12}H_7 & Q_{13}H_7 \\ & & & & Q_{22}H_7 & Q_{23}H_7 \\ \text{Symmetric} & & & & & Q_{33}H_7 \end{bmatrix}_{L\text{th layer}}$$

$$\mathscr{D}_s = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H & Q_{45}H & 0 & 0 \\ & Q_{44}H & 0 & 0 \\ & & Q_{55}H^* & Q_{45}H^* \\ \text{Symmetric} & & & Q_{44}H^* \end{bmatrix}_{L\text{th layer}} \tag{15}$$

where

$$H_i = \frac{1}{i}(h_L^i - h_{L-1}^i), \quad i = 1, 3, 5, 7$$

$$H = \left(H_1 - H_3\frac{4}{h^2}\right), \quad H^* = \left(H_5 - H_3\frac{h^2}{4}\right).$$

The shear rigidity matrix $\mathscr{D}_s$ given by eqn (15) is evolved by incorporating an alternate form of conditions (3), namely

$$\Phi_y + \frac{h^2}{4}\Phi_y^* = 0$$

$$\Phi_x + \frac{h^2}{4}\Phi_x^* = 0 \tag{16}$$

in it and the resulting theory, higher-order shear deformation theory satisfying zero transverse shear conditions on top and bottom bounding planes of the plate (HOST1), becomes consistent in the sense that it satisfies zero transverse shear stress conditions on the top and bottom boundary planes of the plate. If the conditions, given by eqns (16), are not incorporated, the resulting non-consistent theory, higher-order shear deformation theory without satisfying above referred zero transverse shear conditions (HOST2), does not satisfy the zero transverse shear stress conditions on the top and bottom boundary planes of the plate. In this case the shear rigidity matrix $\mathcal{D}_s'$ is defined as

$$\mathcal{D}_s' = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H_1 & Q_{45}H_1 & Q_{55}H_3 & Q_{45}H_3 \\ & Q_{44}H_1 & Q_{45}H_3 & Q_{44}H_3 \\ & & Q_{55}H_5 & Q_{45}H_5 \\ \text{Symmetric} & & & Q_{44}H_5 \end{bmatrix}^{L\text{th layer}} \tag{17}$$

The transverse shear stresses τ_{xz}^L and τ_{yz}^L are not evaluated from eqn (8) as the continuity conditions at the interfaces of the face sheet and the core are not satisfied. For this reason the interlaminar shear $(\tau_{xz}^L, \tau_{yz}^L)$ between layer (L) and layer $(L+1)$ at $z = h_L$ are obtained by integrating the equilibrium equations of elasticity for each layer over the lamina thickness and summing over layers L through n as follows:

$$\tau_{xz}^L|_{z=h_L} = -\sum_{i=1}^{L}\int_{h_{i-1}}^{h_i}\left(\frac{\partial\sigma_x^i}{\partial x} + \frac{\partial\tau_{xy}^i}{\partial y}\right)\mathrm{d}z$$

$$\tau_{yz}^L|_{z=h_L} = -\sum_{i=1}^{L}\int_{h_{i-1}}^{h_i}\left(\frac{\partial\sigma_y^i}{\partial y} + \frac{\partial\tau_{xy}^i}{\partial x}\right)\mathrm{d}z. \tag{18}$$

Substitution of stresses in terms of midplane strains using relations (8) and (4), the integrals of eqns (18) lead to the following expressions for interlaminar shear stresses:

$$\begin{aligned}\tau_{xz}^L|_{z=h_L} = -\sum_{i=1}^{L}\Bigg\{&Q_{11}^i\left(H_2\frac{\partial^2\theta_x}{\partial x^2} + H_4\frac{\partial^2\theta_x^*}{\partial x^2}\right)\\ &+Q_{12}^i\left(H_2\frac{\partial^2\theta_y}{\partial x\,\partial y} + H_4\frac{\partial^2\theta_y^*}{\partial x\,\partial y}\right)\\ &+Q_{13}^i\left(2H_2\frac{\partial^2\theta_x}{\partial x\,\partial y} + H_2\frac{\partial^2\theta_y}{\partial x^2} + 2H_4\frac{\partial^2\theta_x^*}{\partial x\,\partial y} + H_4\frac{\partial^2\theta_y^*}{\partial x^2}\right)\\ &+Q_{23}^i\left(H_2\frac{\partial^2\theta_y}{\partial y^2} + H_4\frac{\partial^2\theta_y^*}{\partial y^2}\right)\\ &+Q_{33}^i\left(H_2\frac{\partial^2\theta_x}{\partial y^2} + H_2\frac{\partial^2\theta_y}{\partial x\,\partial y} + H_4\frac{\partial^2\theta_x^*}{\partial y^2} + H_4\frac{\partial^2\theta_y^*}{\partial x\,\partial y}\right)\Bigg\}\end{aligned}$$

$$\tau_{yz}^L|_{z=-h_L} = -\sum_{i=1}^{L}\left\{Q_{12}^i\left(H_2\frac{\partial^2\theta_x}{\partial x\,\partial y}+H_4\frac{\partial^2\theta_x^*}{\partial x\,\partial y}\right)\right.$$

$$+Q_{22}^i\left(H_2\frac{\partial^2\theta_y}{\partial y^2}+H_4\frac{\partial^2\theta_y^*}{\partial y^2}\right)$$

$$+Q_{23}^i\left(H_2\frac{\partial^2\theta_x}{\partial y^2}+2H_2\frac{\partial^2\theta_y}{\partial x\,\partial y}+H_4\frac{\partial^2\theta_x^*}{\partial y^2}+2H_4\frac{\partial^2\theta_y^*}{\partial x\,\partial y}\right)$$

$$+Q_{13}^i\left(H_2\frac{\partial^2\theta_x}{\partial x^2}+H_4\frac{\partial^2\theta_x^*}{\partial x^2}\right)$$

$$\left.+Q_{33}^i\left(H_2\frac{\partial^2\theta_x}{\partial x\,\partial y}+H_2\frac{\partial^2\theta_y}{\partial x^2}+H_4\frac{\partial^2\theta_x^*}{\partial x\,\partial y}+H_4\frac{\partial^2\theta_y^*}{\partial x^2}\right)\right\} \quad (19)$$

in which, H_2, H_4 and Q_{ij} have already been defined.

3. FINITE ELEMENT FORMULATION

In the standard finite element technique, the total solution domain is discretized into NE subdomains (elements) such that

$$\pi(\delta) = \sum_{e=1}^{NE}\pi^e(\delta) \quad (20)$$

where π and π^e are the total potential of the system and the element, respectively. The element potential can be expressed in terms of internal strain energy U^e and the external work done W^e for an element "e" as

$$\pi^e(\delta) = U^e - W^e \quad (21)$$

in which $\boldsymbol{\delta}$ is the vector of unknown displacement variables in the problem and it is defined by eqn (11). If the same interpolation function is used to define all the components of the generalized displacement vector $\boldsymbol{\delta}$, we can write

$$\boldsymbol{\delta} = \sum_{i=1}^{NN} N_i\boldsymbol{\delta}_i \quad (22)$$

in which N_i is the interpolating (shape) function associated with node i, $\boldsymbol{\delta}_i$ the value of $\boldsymbol{\delta}$ corresponding to node i and NN the number of nodes in an element.

The bending curvatures ($\boldsymbol{\chi}$, $\boldsymbol{\chi}^*$) and the transverse shear strains ($\boldsymbol{\Phi}$, $\boldsymbol{\Phi}^*$) are written in terms of the degrees of freedom $\boldsymbol{\delta}$ by making use of eqns (5) as follows:

$$\begin{Bmatrix}\boldsymbol{\chi}\\ \boldsymbol{\chi}^*\end{Bmatrix} = \mathscr{L}_b\boldsymbol{\delta}$$

$$\begin{Bmatrix}\boldsymbol{\Phi}\\ \boldsymbol{\Phi}^*\end{Bmatrix} = \mathscr{L}_s\boldsymbol{\delta}. \quad (23)$$

Subscripts b and s refer to bending and shear, respectively, and matrices $\mathscr{L}_b$ and $\mathscr{L}_s$ are defined as follows:

$$\mathscr{L}_b = \begin{bmatrix} 0 & \partial/\partial x & 0 & 0 & 0 \\ 0 & 0 & \partial/\partial y & 0 & 0 \\ 0 & \partial/\partial y & \partial/\partial x & 0 & 0 \\ 0 & 0 & 0 & \partial/\partial x & 0 \\ 0 & 0 & 0 & 0 & \partial/\partial y \\ 0 & 0 & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

$$\mathscr{L}_s = \begin{bmatrix} \partial/\partial x & 1 & 0 & 0 & 0 \\ \partial/\partial y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{24}$$

With the generalized displacement vector $\boldsymbol{\delta}$ known at all points within the element, the generalized strain vectors at any point are determined with the aid of eqns (24) and (22) as follows:

$$\begin{Bmatrix} \chi \\ \chi^* \end{Bmatrix} = \mathscr{L}_b \boldsymbol{\delta} = \mathscr{L}_b \sum_{i=1}^{NN} N_i \boldsymbol{\delta}_i = \sum_{i=1}^{NN} \mathscr{B}_{ib} \boldsymbol{\delta}_i = \mathscr{B}_b \mathbf{d}$$

$$\begin{Bmatrix} \Phi \\ \Phi^* \end{Bmatrix} = \mathscr{L}_s \boldsymbol{\delta} = \mathscr{L}_s \sum_{i=1}^{NN} N_i \boldsymbol{\delta}_i = \sum_{i=1}^{NN} \mathscr{B}_{is} \boldsymbol{\delta}_i = \mathscr{B}_s \mathbf{d} \tag{25a}$$

where

$$\mathscr{B}_{ib} = \mathscr{L}_b N_i, \quad \mathscr{B}_b = [\mathscr{B}_{1b} \,\vdots\, \mathscr{B}_{2b} \,\vdots\, \ldots \,\vdots\, \mathscr{B}_{NNb}]$$

$$\mathscr{B}_{is} = \mathscr{L}_s N_i, \quad \mathscr{B}_s = [\mathscr{B}_{1s} \,\vdots\, \mathscr{B}_{2s} \,\vdots\, \ldots \,\vdots\, \mathscr{B}_{NNs}]$$

and

$$\mathbf{d}' = \{\boldsymbol{\delta}'_1, \boldsymbol{\delta}'_2, \ldots, \boldsymbol{\delta}'_{NN}\}. \tag{25b}$$

For the elastostatic analysis, the internal strain energy of an element due to bending and shear can be determined by integrating the products of moment stress-resultants and bending curvatures, and shear stress-resultants and shear strains over the area of an element

$$U^e = \frac{1}{2} \int_A \left[(\chi', \chi^{*\prime}) \begin{Bmatrix} \mathbf{M} \\ \mathbf{M}^* \end{Bmatrix} + (\Phi', \Phi^{*\prime}) \begin{Bmatrix} \mathbf{Q} \\ \mathbf{Q}^* \end{Bmatrix} \right] \mathrm{d}A. \tag{26}$$

Implementing the stress resultants given by eqn (13) in the strain energy expression (26), we obtain

$$U^e = \frac{1}{2} \int_A \left[(\chi', \chi^{*\prime}) \mathscr{D}_b \begin{Bmatrix} \chi \\ \chi^* \end{Bmatrix} + (\Phi', \Phi^{*\prime}) \mathscr{D}_s \begin{Bmatrix} \Phi \\ \Phi^* \end{Bmatrix} \right] \mathrm{d}A. \tag{27}$$

Substitution of eqn (25a) for bending and shear strains into eqn (27) leads to the strain energy expression in terms of the nodal displacements which is given as follows:

$$U^e = \frac{1}{2} \int_A \{\mathbf{d}'(\mathscr{B}'_b \mathscr{D}_b \mathscr{B}_b)\mathbf{d} + \mathbf{d}'(\mathscr{B}'_s \mathscr{D}_s \mathscr{B}_s)\mathbf{d}\} \, \mathrm{d}A. \tag{28}$$

This can be written in a concise form as

$$U^e = \tfrac{1}{2}[\mathbf{d}^t \quad \mathscr{K}^e \quad \mathbf{d}] \tag{29}$$

in which $\mathscr{K}^e$ is the stiffness matrix for an element "e" which includes bending and the transverse shear effects and is given by

$$\mathscr{K}^e = \int_A \{\mathscr{B}_b^t \mathscr{D}_b \mathscr{B}_b + \mathscr{B}_s^t \mathscr{D}_s \mathscr{B}_s\}\,\mathrm{d}A. \tag{30}$$

The computation of the element stiffness matrix from eqn (30) is economized by explicit multiplication of the $\mathscr{B}_i$, $\mathscr{D}$ and $\mathscr{B}_j$ matrices instead of carrying out the full matrix multiplication of the triple product. In addition, due to symmetry of the stiffness matrix, only the blocks $\mathscr{K}_{ij}$ lying on one side of the main diagonal are formed. The integral is evaluated using the Gauss quadrature

$$\mathscr{K}_{ij}^e = \int_{-1}^{1}\int_{-1}^{1} \mathscr{B}_i^t \mathscr{D} \mathscr{B}_j |\mathscr{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\mathscr{K}_{ij}^e = \sum_{a=1}^{g}\sum_{b=1}^{g} W_a W_b |\mathscr{J}| \mathscr{B}_i^t \mathscr{D} \mathscr{B}_j \tag{31}$$

in which W_a and W_b are weighting coefficients, g the number of numerical quadrature points in each of the two directions (x and y) and $|\mathscr{J}|$ the determinant of the standard Jacobian matrix. Subscripts i and j vary from 1 to a number of nodes per element (NN). Matrix $\mathscr{D}$ is defined by eqn (13) and matrices $\mathscr{B}_i$ and $\mathscr{B}_j$ are given by

$$\mathscr{B}_i = \begin{bmatrix} \mathscr{B}_{ib} \\ \mathscr{B}_{is} \end{bmatrix} \quad \text{and} \quad \mathscr{B}_j = \begin{bmatrix} \mathscr{B}_{jb} \\ \mathscr{B}_{js} \end{bmatrix}. \tag{32}$$

For the problem of bending of sandwich plates, the applied external forces may consist of concentrated nodal loads F_c, each corresponding to nodal degrees of freedom, a distributed load q acting over the element in the z-direction and a sinusoidal distributed load P_{mn} acting over the element in the z-direction. The total external work done by these forces may be expressed as follows:

$$W^e = \mathbf{d}^t\mathbf{F}_c + \mathbf{d}^t \int_A \{N_1, 0, 0, 0, 0, N_2, 0, 0, 0, 0, N_3, \ldots, N_{NN}, 0, 0, 0, 0\}^t (q + P_{mn})\,\mathrm{d}A. \tag{33}$$

The integral in eqn (33) is evaluated numerically using Gauss quadrature as follows:

$$P = \sum_{a=1}^{g}\sum_{b=1}^{g} W_a W_b |\mathscr{J}| \{N_1, 0, 0, 0, 0, N_2, 0, 0, 0, 0, \ldots, N_{NN}, 0, 0, 0, 0\}^t \times \left\{q + p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\right\} \tag{34}$$

in which a and b are the plate dimensions; x and y are the Gauss point coordinates and m and n are the usual harmonic numbers.

4. NUMERICAL EXAMPLES AND DISCUSSION

Validity of the finite element formulations of the higher-order theories is established by comparing results for laminated and sandwich plate problems with those available in the form of exact, closed form and other finite element solutions. The element properties in the isoparametric finite element formulation presented here are evaluated through Gauss quadrature. The selective integration scheme, namely 3×3 for flexure and 2×2 for shear contributions, has been employed. The geometrical and material properties for two different composite plate problems are as follows.

Material I

$$C_{11} = 0.999781; \qquad C_{33} = 0.262931$$
$$C_{12} = C_{21} = 0.231192; \quad C_{44} = 0.266810$$
$$C_{22} = 0.524886; \qquad C_{55} = 0.159914$$
$$h_1 = 0.01,\ h_2 = 0.08,\ h_3 = 0.01,\ \alpha = 0^\circ, q = 1. \tag{35}$$

Material II

Face sheets

$$\frac{E_1}{E_2} = 25; \quad \frac{G_{12}}{E_2} = 0.5; \quad \frac{G_{23}}{E_2} = 0.2$$
$$E_2 = 10^6,\quad G_{13} = G_{12},\quad \nu_{12} = 0.25$$
$$h_1 = h_3 = 0.1\,h,\quad \alpha = 0^\circ,\quad p_{mn} = 1.$$

Core

$$E_x = E_y = 0.4\times 10^5;\quad G_{xz} = G_{yz} = 0.6\times 10^5$$
$$G_{xy} = 0.16\times 10^5;\quad \nu_{xy} = 0.25,\quad h_2 = 0.8h$$
$$\nu_{21} = \frac{E_2}{E_1}\nu_{12};\quad \text{directions 1 and } x \text{ are coincident.} \tag{36}$$

In both the examples that follow, the plate is square and simply supported along all four edges. Except for the convergence study the plate is discretized with four, 9 noded quadrilateral elements in a quarter plate. The finite element evaluations of stresses are at the nearest Gauss points. The deflection and stresses presented here are nondimensionalized using the following multipliers:

$$m_1 = \frac{100h^3E_2}{p_{mn}a^4};\quad m_2 = \frac{h^2}{p_{mn}a^2};\quad m_3 = \frac{h}{p_{mn}a};\quad m_4 = \frac{1}{q};\quad m_5 = \frac{C_{11}\,(\text{core})}{hq}. \tag{37}$$

Superscripts "e" and "c" used in Tables 1–8 represent stress predictions from equilibrium and constitutive relations, respectively. The two examples considered are described below.

4.1. *Example* 1 : *symmetric laminated plate under uniform transverse pressure*

This example is selected from Srinivas and Rao (1970). The set of material and geometrical properties given by relations (35) are used. The full (6×6) material stiffness matrix given in Srinivas and Rao (1970) is reduced (5×5) to suit the present theories, by assuming $\sigma_z = 0$ and eliminating ε_z from the stress–strain constitutive relations. The final material stiffness coefficients adopted are given by relations (35). All the stiffness matrix coefficients for top and bottom laminae are some constant multiplier (modular ratio, R) times the corresponding stiffness matrix coefficients for the middle lamina. The numerical results showing convergence of deflection and stresses with mesh refinement are given in Table 1. The convergence of transverse shear stress value with mesh refinement is shown in Fig. 2. The transverse deflection and stresses at different locations in the thickness direction and for various modular ratios (R = 5, 10, 15, 25, 50, 100) are given in Tables 2–5. The effect of varying modular ratio (R) on transverse deflection is shown in Fig. 3. The effect of modular ratio on inplane normal stresses in the x- and y-directions at z = 0.05

Table 1. Convergence of maximum stresses and displacement in a simply supported square laminated plate (material I, $a/h = 10$, $R = 5$)

Source	Mesh size in quarter plate	$\sigma_{x1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{y1} \times m_4$ $(a/2, a/2, h/2)$	$\tau_{xy1} \times m_4$ $(0, 0, h/2)$	$\tau^e_{xz2} \times m_4$ $(0, a/2, 0)$	$\tau^e_{yz2} \times m_4$ $(a/2, 0, 0)$	$w_0 \times m_5$ $(a/2, a/2, 0)$
HOST1	2×2	62.38	38.93	−33.22	3.089	2.541	256.13
	3×3	60.31	38.43	−34.08	3.652	2.874	256.47
	4×4	60.54	38.57	−33.98	3.832	3.069	256.38
	5×5	60.35	38.26	−34.41	3.954	3.179	256.43
HOST2	2×2	61.03	38.78	−33.81	3.259	2.539	257.78
	3×3	60.65	38.58	−34.35	3.634	2.879	257.44
	4×4	60.55	38.53	−34.57	3.833	3.068	257.38
	5×5	60.52	38.52	−34.69	3.953	3.188	257.37
Srinivas and Rao (1970)	—	60.353	38.491	—	4.3641	—	258.97
CLT	—	61.141	36.622	—	4.5899	—	216.94

is shown in Figs 4 and 5, respectively. The following general observations are made from the results presented in Tables 1 – 5 and Figs 2–5.

(1) Deflection and inplane stresses can be accurately predicted without refining the mesh, as the 2 × 2 mesh in a quarter plate gives sufficiently accurate results. The refined mesh (5 × 5 in a quarter plate or more) is necessary for accurate prediction of transverse shear stresses.

(2) Errors in stress and deflection predictions increase with increasing value of modular ratio (R). The differences in the first (FOST) and higher-order shear deformation theories (HOST1, HOST2) are very high for a large value of modular ratio, say $R = 100$.

(3) CPT and FOST underpredict deflections considerably. Deflections obtained using higher-order theories agree well with exact solutions.

(4) Out of the two higher-order shear deformation theories presented, the one which does not satisfy free transverse shear stress conditions on top and bottom boundary planes of the plate (HOST2) is preferred as its agreement with exact solutions is superior than the other one (HOST1).

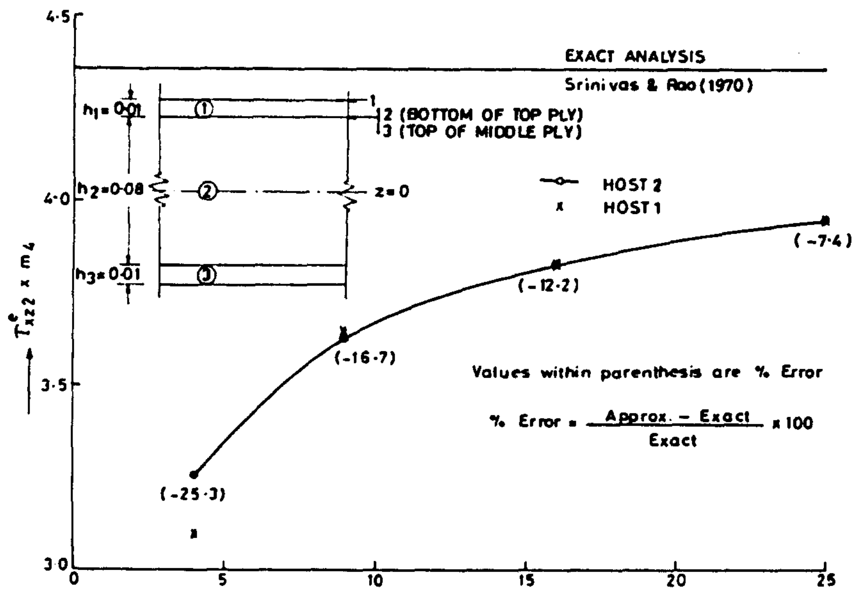

Fig. 2. Convergence of transverse shear stress with the mesh refinement for a simply supported square laminated plate under uniform transverse load ($a/h = 10$).

Table 2. Maximum stresses and displacement in a simply supported square laminated plate (material I, $a/h = 10$, $R = 5$)

Source	$\sigma_{x1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{x2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{x3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\sigma_{y1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{y2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{y3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\tau^e_{xz1} \times m_4$ $(0, a/2, 4h/10)$	$\tau^e_{xz2} \times m_4$ $(0, a/2, 0)$	$\tau^e_{xz3} \times m_4$ $(0, a/2, -4h/10)$	$w_0 \times m_5$ $(a/2, a/2, 0)$
HOST1	62.38 (3.36)	46.91 (0.62)	9.382 (0.45)	38.93 (1.14)	30.33 (0.77)	6.065 (−1.56)	2.566 (−31.0)	3.089 (−29.2)	2.566 (−21.5)	256.13 (−1.1)
HOST2	61.03 (1.12)	47.32 (1.49)	9.463 (1.32)	38.78 (0.75)	30.42 (1.07)	6.083 (−1.27)	2.422 (−34.9)	3.259 (−25.3)	2.422 (−25.9)	257.78 (−0.46)
FOST	61.87 (2.51)	49.50 (6.17)	9.899 (5.99)	36.65 (−4.78)	29.32 (−2.58)	5.864 (−4.82)	2.444 (−34.29)	3.313 (−24.1)	2.444 (−25.2)	236.10 (−8.83)
Srinivas and Rao (1970)	60.353	46.623	9.340	38.491	30.097	6.161	3.7194	4.3641	3.2675	258.97
CLT	61.141 (1.31)	48.913 (4.91)	9.783 (4.74)	36.622 (−4.86)	29.297 (−2.66)	5.860 (−4.89)	3.3860 (−8.96)	4.5899 (5.17)	3.3860 (3.63)	216.94 (−16.23)

Table 3. Maximum stresses and displacement in a simply supported square laminated plate (material I, $a/h = 10$, $R = 10$)

Source	$\sigma_{x1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{x2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{x3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\sigma_{y1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{y2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{y3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\tau^e_{xz1} \times m_4$ $(0, a/2, 4h/10)$	$\tau^e_{xz2} \times m_4$ $(0, a/2, 0)$	$\tau^e_{xz3} \times m_4$ $(0, a/2, -4h/10)$	$w_0 \times m_5$ $(a/2, a/2, 0)$
HOST1	64.65 (−1.04)	51.31 (5.02)	5.131 (4.65)	42.83 (−1.69)	33.97 (1.67)	3.397 (−2.94)	2.587 (−34.1)	3.147 (−23.2)	2.587 (−26.4)	152.33 (−4.42)
HOST2	66.23 (1.37)	50.00 (2.34)	5.000 (1.98)	43.78 (0.49)	33.81 (1.19)	3.381 (−3.4)	2.629 (−33.1)	3.073 (−25.0	2.629 (−25.2)	156.18 (−2.01)
FOST	67.80 (3.78)	54.24 (11.02)	5.424 (10.63)	40.10 (−7.96)	32.08 (−3.99)	3.208 (−8.34)	2.676 (−31.9)	3.152 (−23.0)	2.676 (−23.9)	131.095 (−17.75)
Srinivas and Rao (1970)	65.332	48.857	4.903	43.566	33.413	3.500	3.9285	4.0959	3.5154	159.38
CLT	66.947 (2.47)	53.557 (9.62)	5.356 (9.24)	40.099 (−7.96)	32.079 (−3.99)	3.208 (−8.34)	3.7075 (−5.63)	4.3666 (6.61)	3.7075 (5.46)	118.77 (−25.48)

Table 4. Maximum stresses and displacement in a simply supported square laminated plate (material I, $a/h = 10$, $R = 15$)

Source	$\sigma_{x1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{x2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{x3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\sigma_{y1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{y2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{y3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\tau^e_{xz1} \times m_4$ $(0, a/2, 4h/10)$	$\tau^e_{xz2} \times m_4$ $(0, a/2, 0)$	$\tau^e_{xz3} \times m_4$ $(0, a/2, -4h/10)$	$w_0 \times m_5$ $(a/2, a/2, 0)$
HOST1	66.62 (−0.25)	51.97 (7.60)	3.465 (7.01)	44.92 (−3.24)	35.41 (1.30)	2.361 (−5.33)	2.691 (−31.98)	3.035 (−23.43)	2.691 (−24.77)	110.43 (−9.28)
HOST2	67.88 (1.64)	49.94 (3.40)	3.329 (2.81)	46.45 (0.06)	35.36 (1.16)	2.357 (−5.49)	2.693 (−31.92)	2.989 (−24.59)	2.693 (−24.71)	117.14 (−3.76)
FOST	70.04 (4.87)	56.03 (16.00)	3.735 (15.35)	41.39 (−10.84)	33.11 (−5.28)	2.208 (−11.47)	2.764 (−30.13)	3.091 (−22.02)	2.764 (−22.72)	90.85 (−25.36)
Srinivas and Rao (1970)	66.787	48.299	3.238	46.424	34.955	2.494	3.9559	3.9638	3.5768	121.72
CLT	69.135 (3.52)	55.308 (14.51)	3.687 (13.87)	41.410 (−10.80)	33.128 (−5.23)	2.209 (−11.43)	3.8287 (−3.22)	4.2825 (8.04)	3.8287 (7.04)	81.768 (−32.82)

Table 5. Maximum stresses and displacement in a simply supported square laminated plate (material I, $a/h = 10$)

Source	R	$\sigma_{x1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{x2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{x3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\sigma_{y1} \times m_4$ $(a/2, a/2, h/2)$	$\sigma_{y2} \times m_4$ $(a/2, a/2, 4h/10)$ in face sheet	$\sigma_{y3} \times m_4$ $(a/2, a/2, 4h/10)$ in core	$\tau^e_{xz1} \times m_4$ $(0, a/2, 4h/10)$	$\tau^e_{xz2} \times m_4$ $(0, a/2, 0)$	$w_0 \times m_5$ $(a/2, a/2, 0)$
HOST1		66.66	53.03	2.121	46.64	37.06	1.482	2.744	2.973	72.748
HOST2	25	68.89	48.27	1.931	49.88	37.03	1.481	2.726	2.897	82.86
FOST		71.94	57.55	2.302	42.49	33.99	1.36	2.838	3.040	56.331
HOST1		67.37	52.75	1.055	48.54	38.39	0.7678	2.791	2.898	39.873
HOST2	50	69.14	43.57	0.8714	55.04	38.89	0.7779	2.708	2.782	53.301
FOST		73.44	58.75	1.175	43.35	34.68	0.6935	2.897	3.000	28.904
HOST1		67.30	52.57	0.5257	49.54	39.33	0.3933	2.808	2.861	21.166
HOST2	100	69.18	37.15	0.3715	60.63	40.15	0.4015	2.650	2.677	34.521
FOST		74.22	59.37	0.5937	43.79	35.03	0.3503	2.927	2.979	14.647

Table 6. Maximum stresses and displacement in a simply supported square sandwich plate (material II, $a/h = 4$)

Source	$\sigma_x \times m_2$ $(a/2, a/2, h/2)$	$\sigma_x \times m_2$ $(a/2, a/2, 4h/10)$	$\sigma_y \times m_2$ $(a/2, a/2, h/2)$	$\tau_{xy} \times m_2$ $(0, 0, h/2)$	$\tau^e_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^c_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^e_{yz} \times m_3$ $(a/2, 0, 0)$	$\tau^c_{yz} \times m_3$ $(a/2, 0, 0)$	$w_0 \times m_1$ $(a/2, a/2, 0)$
HOST1	1.2470 (−19.9)	0.2416	0.2338 (−9.9)	−0.1343 (−6.5)	0.2245 (−6.1)	0.2382	0.08653 (−19.3)	0.1132	0.6947
HOST2	1.5230 (−2.1)	−0.0120	0.2414 (−7.0)	−0.1419 (−1.3)	0.2200 (−7.9)	0.2750	0.08898 (−17.0)	0.1137	0.7160
FOST	0.9056 (−41.8)	0.7244	0.1578 (−39.2)	−0.0912 (−36.5)	0.2505 (4.8)	0.0995	0.06603 (−38.4)	0.0436	0.4755
Pagano (1970)	1.556	−0.2330	0.2595	−0.1437	0.2390	—	0.1072	—	—
Reddy and Chao (1981)—FEM	0.8650	—	0.1517	−0.0878	—	0.0994	—	0.1740	0.4761
Reddy and Chao (1981)—CFS	0.8670	—	0.1520	−0.0877	—	0.0993	—	0.1740	0.4767
CLT	1.097 (−29.5)	0.878	0.0543 (−79.1)	−0.0433 (−69.9)	0.324 (35.6)	—	0.0295 (−72.5)	—	—

Table 7. Maximum stresses and displacement in a simply supported square sandwich plate (material II, $a/h = 10$)

Source	$\sigma_x \times m_2$ $(a/2, a/2, h/2)$	$\sigma_x \times m_2$ $(a/2, a/2, 4h/10)$	$\sigma_y \times m_2$ $(a/2, a/2, h/2)$	$\tau_{xy} \times m_2$ $(0, 0, h/2)$	$\tau^e_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^c_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^e_{yz} \times m_3$ $(a/2, 0, 0)$	$\tau^c_{yz} \times m_3$ $(a/2, 0, 0)$	$w_0 \times m_1$ $(a/2, a/2, 0)$
HOST1	1.110 (−3.7)	0.7445 (18.6)	0.1017 (−7.9)	−0.0666 (−5.8)	0.2700 (−10.0)	0.2841	0.04366 (−17.2)	0.05593	0.2023
HOST2	1.166 (1.1)	0.6878 (9.5)	0.1052 (−4.7)	−0.0692 (−2.1)	0.2685 (−10.5)	0.3400	0.04462 (−15.3)	0.05642	0.2087
FOST	1.062 (−7.9)	0.8495 (35.3)	0.08057 (−27.0)	−0.05532 (−21.8)	0.2779 (−7.4)	0.1112	0.03636 (−31.0)	0.02384	0.1557
Pagano (1970)	1.153	0.628	0.1104	−0.0707	0.3000	—	0.05270	—	—
Reddy and Chao (1981)—FEM	1.015	—	0.0774	−0.0535	—	0.1112	—	0.095	0.1558
Reddy and Chao (1981)—CFS	1.017	—	0.0776	−0.0533	—	0.1110	—	0.095	0.1560
CLT	1.097 (−4.9)	0.878 (39.8)	0.0543 (−50.8)	−0.0433 (−38.8)	0.324 (8.0)	—	0.0295 (−44.0)	—	—

Table 8. Maximum stresses and displacement in a simply supported square sandwich plate (material II, $a/h = 100$)

Source	$\sigma_x \times m_2$ $(a/2, a/2, h/2)$	$\sigma_x \times m_2$ $(a/2, a/2, 4h/10)$	$\sigma_y \times m_2$ $(a/2, a/2, h/2)$	$\tau_{xy} \times m_2$ $(0, 0, h/2)$	$\tau^e_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^c_{xz} \times m_3$ $(0, a/2, 0)$	$\tau^e_{yz} \times m_3$ $(a/2, 0, 0)$	$\tau^c_{yz} \times m_3$ $(a/2, 0, 0)$	$w_0 \times m_1$ $(a/2, a/2, 0)$
HOST1	1.108 (0.9)	0.8852 (1.2)	0.0554 (0.7)	−0.0440 (0.7)	0.2880 (−11.1)	0.3001	0.02703 (−9.0)	0.03362	0.0891
HOST2	1.109 (1.0)	0.8847 (1.1)	0.0554 (0.7)	−0.0440 (0.7)	0.2880 (−11.1)	0.3627	0.02704 (−9.0)	0.03322	0.0891
FOST	1.104 (0.5)	0.8836 (1.0)	0.0546 (−0.7)	−0.0435 (−0.5)	0.2875 (−11.3)	0.1152	0.02695 (−9.3)	0.01767	0.0883
Pagano (1970)	1.098	0.875	0.0550	−0.0437	0.3240	—	0.02970	—	—
Reddy and Chao (1981)—FEM	1.063	—	0.0530	−0.0421	—	0.1158	—	0.072	0.0882
Reddy and Chao (1981)—CFS	1.067	—	0.0531	−0.0420	—	0.1149	—	0.069	0.0885
CLT	1.097 (−0.1)	0.878 (0.3)	0.0543 (−1.3)	−0.0433 (−0.9)	0.3240 (0.0)	—	0.02950 (−0.7)	—	—

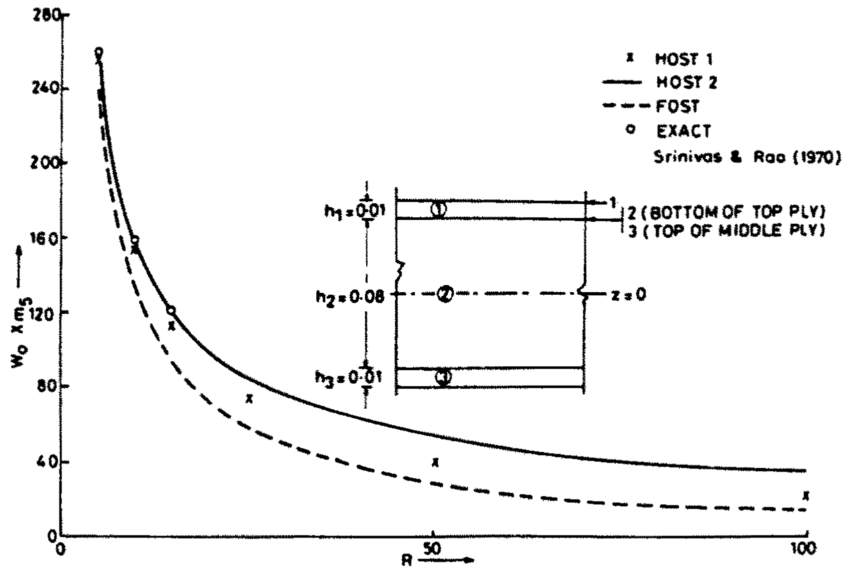

Fig. 3. Effect of modular ratio (top or bottom/middle) on maximum transverse deflection for a simply supported, symmetrically laminated, square plate under uniform transverse load ($a/h = 10$).

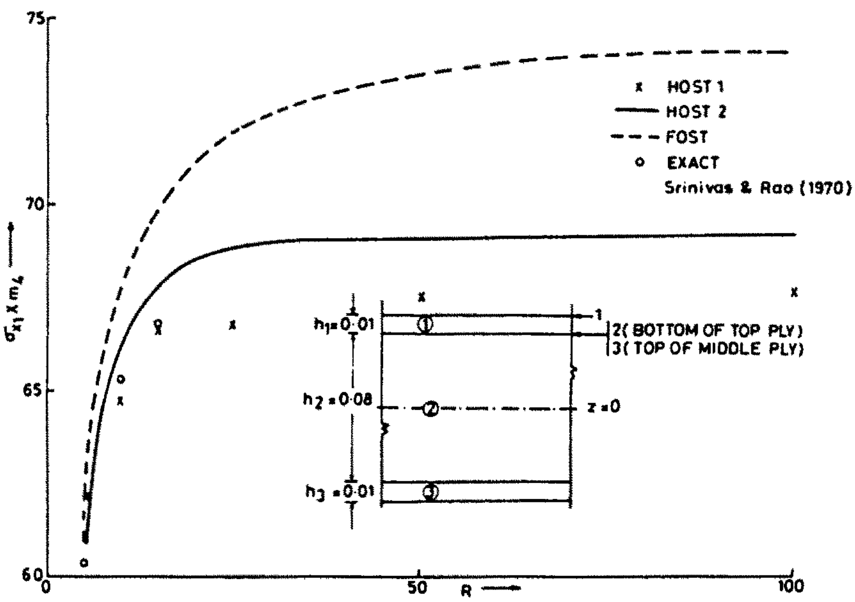

Fig. 4. Effect of modular ratio (top or bottom/middle) on maximum inplane normal stress (at level 1 in x-direction) for a simply supported, symmetrically laminated square plate under uniform transverse load ($a/h = 10$).

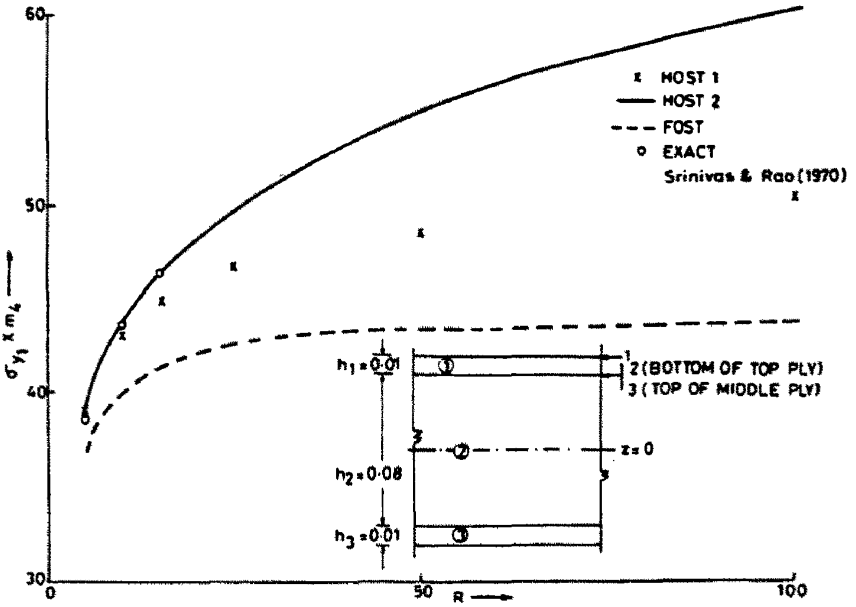

Fig. 5. Effect of modular ratio (top or bottom/middle) on maximum inplane normal stress (at level 1 in y-direction) for a simply supported, symmetrically laminated square plate under uniform transverse load (a/h = 10).

4.2. *Example 2: sandwich plate under sinusoidal distributed load*

This example is selected from Pagano (1970). The properties given by relations (36) are used for the analysis. The elastic properties given by Pagano (1970) are modified accordingly by introducing therein the assumption of $\sigma_z = 0$. The results for deflection and stresses with percentage errors specified within parentheses for $a/h = 4$, 10 and 100 are presented in Tables 6–8, respectively. The effect of plate side-to-thickness ratio on transverse deflection is shown in Fig. 6. The variation of inplane displacement along the x-direction (u) through the plate thickness is shown in Fig. 7. The effect of plate side-to-thickness ratio on transverse shear stresses (τ_{yz}) and inplane normal stresses (σ_x) are shown in Figs 8 and

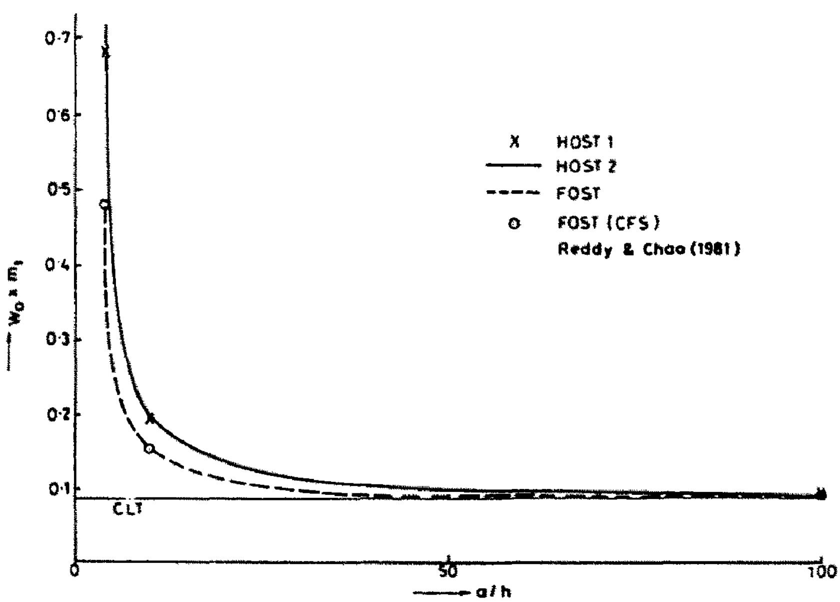

Fig. 6. Effect of plate side-to-thickness ratios on the transverse deflections for a simply supported square sandwich plate under sinusoidal transverse load.

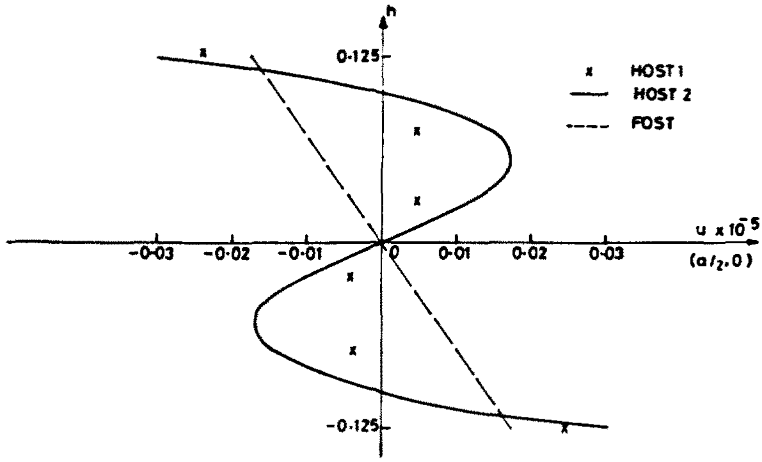

Fig. 7. Variation of inplane displacement along x-axis for a simply supported square sandwich plate ($a/h = 4$) under sinusoidal transverse load.

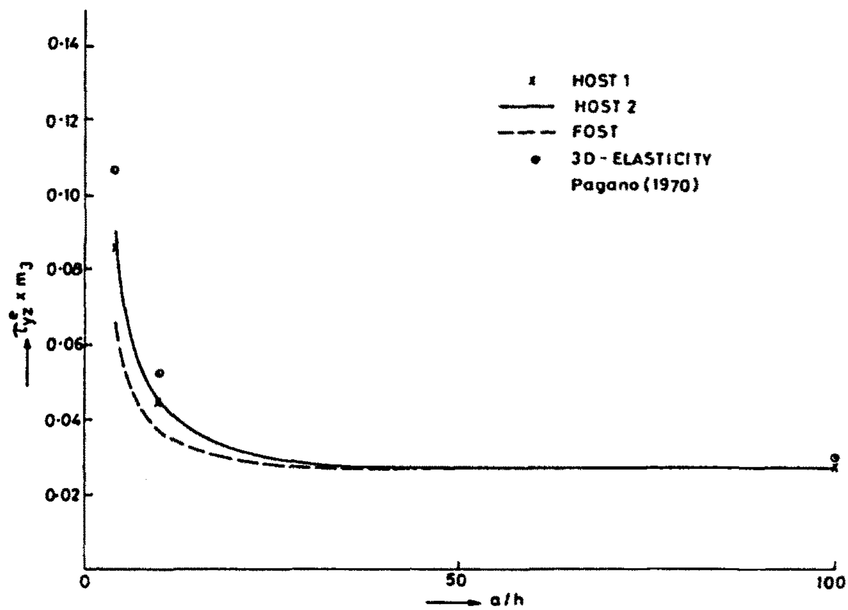

Fig. 8. Effect of plate side-to-thickness ratios on the transverse shear stresses for a simply supported square sandwich plate under sinusoidal transverse load.

9, respectively. The following observations are made from the results presented in Tables 6–8 and Figs 6–9.

(1) For thick ($a/h = 4$) and moderately thick ($a/h = 10$) plates, the deflection and stresses predicted by CPT and FOST are grossly in error.

(2) All the theories agree well with each other for thin plates ($a/h = 100$).

(3) The transverse cross-section warping phenomenon which will be predominant for a thick sandwich plate is evident in the present higher-order theories (Fig. 7).

(4) The first and the last observations made in Example 1 are true for this example too.

5. CONCLUSIONS

The results from the higher-order two-dimensional plate theories developed here compare well with three-dimensional elasticity solutions. The theories lead to realistic parabolic

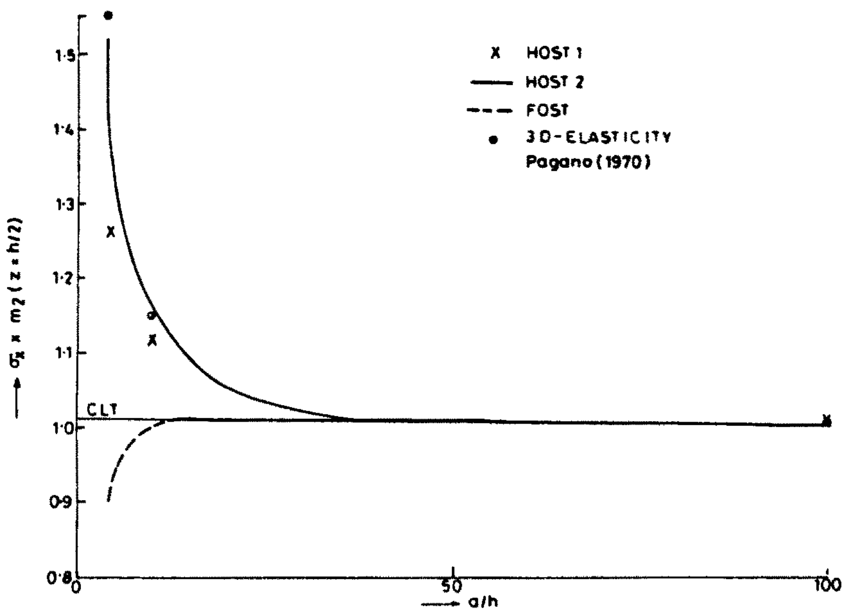

Fig. 9. Effect of plate side-to-thickness ratios on the inplane normal stresses for a simply supported square sandwich plate under sinusoidal transverse load.

variation of transverse shear stresses through the plate thickness, thus they do not require the use of shear correction coefficients. The simplifying assumptions made in CPT and FOST are reflected by high percentage error in the results of thick sandwich or laminated plates with highly stiff facings. It is believed that the improved shear deformation theory presented here is essential for reliable analyses of sandwich type laminated composite plates. Finally, the general isoparametric finite element formulation of these theories presented can be applied to analyse any practical plate structures.

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